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PERIODIC SOLUTIONS TO NONLINEAR ONE DIMENSIONAL WAVE EQUATION WITH X-DEPENDENT COEFFICIENTS

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ABSTRACT. This paper deals with t-periodicity and regularity of solutions to the one dimensional nonlinear wave equation with x-dependent coefficients

1. INTRODUCTION

This paper deals with the study of T-periodic solutions for the nonlinear one dimensional wave equation with x-dependent coefficients:

$$
\begin{aligned}
&u(x)y_{tt} - (u(x)y_x)_x + g(y) = f(x,t), \quad 0 < x < \pi, \quad t \in \mathbb{R},\\
&y(0,t) = y(\pi,t) = 0,\\
&y(x,t+T) = y(x,t), \quad 0 < x < \pi, \quad t \in \mathbb{R},
\end{aligned}
\tag{1.1}
$$

under the following hypotheses:

(H1) $u \in H^2(0,\pi)$; $u(x) \geq 1$, $\forall x \in [0,\pi]$ and

$$\rho = \text{ess inf}\, \eta_u(x) > 0, \tag{1.2}$$

where

$$\eta_u(x) = \frac{1}{2}\frac{u''}{u} - \frac{1}{4}\left(\frac{u'}{u}\right)^2. \tag{1.3}$$

(H2) The function $g : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and

$$|g(y) - g(z)| \leq \gamma |y - z|, \quad y, z \in \mathbb{R}, \tag{1.4}$$

for some $\gamma > 0$.

Another assumption relating f and g is given in Section 3 (Hypothesis (H3)).

Equation (1.1) is a nonlinear model for the forced vibrations of a nonhomogeneous string as well as for propagation of waves in nonisotropic media (see e.g. [2]). More precisely, the displacement $y(z,t)$ at depth z and time t in the case of plane seismic waves is described by the equation

$$\rho(z)y_{tt} - (\mu(z)y_z)_z = 0 \tag{1.5}$$

with some boundary conditions in z and initial conditions in t, where $\rho(z)$ is the rock density at depth z and $\mu(z)$ is the elasticity coefficient at z.

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By a change of variable $z \to x$ given by $x = \int_0^z (\frac{\rho(s)}{\mu(s)})^{1/2} ds$ the equation (1.5) leads to (1.1) (with $f = g = 0$ and $u = (\rho\mu)^{1/2}$—the acoustic impedance function).

An inverse problem for (1.1) was recently studied by the authors in [3] for $g = 0$ and $T = (2k+1)/2p$; $k = 0, 1, \dots$; $p = 1, 2, \dots$. The special case $u \equiv$constant was extensively studied in the last years beginning with the classical paper of P. Rabinowitz [9] and more recently by Brézis and Nirenberg [5], Bahri and Brézis [1]. For a complete reference we refer to the survey of Brézis [4]. In these papers T is a rational multiple of π. The case in which T can be an irrational multiple of π (i.e., $T = \pi\beta$ for some irrational number β) has been investigated by McKenna [8]. (See also [7].) To our best knowledge there are no previous results on the more general case of (1.1) considered here. The main result of this paper is Theorem 3.1, which is an extension of results obtained by Bahri and Brézis [1].

The main notations in this paper are given below:

$Q = (0, \pi) \times (0, T)$

The inner product in $L^2(Q)$ is denoted by $\langle \cdot, \cdot \rangle$ and is defined as

$$\langle f, g \rangle = \int_Q u(x) f(x,t) \bar{g}(x,t)\, dx\, dt, \quad f, g \in L^2(Q). \tag{1.6}$$

Accordingly,

$$|f|^2_{L^2(Q)} = |f|^2 = \int_Q u(x) |f(x,t)|^2 dx\, dt. \tag{1.7}$$

In addition:

$$|f|_{L^\infty(Q)} = |f|_\infty = \text{ess sup}\, \{|f(x,t)|; (x,t) \in Q\}, \quad f \in L^\infty(Q). \tag{1.8}$$

$H^1(Q)$ is the usual Sobolev space, $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ is the set of all positive integers. Throughout this paper T is a rational multiple of π. We will write this as

$$T = 2\pi \frac{p}{q}, \quad \text{with } p, q \text{ relatively prime positive integers.} \tag{1.9}$$

If S is a closed subset of $L^2(Q)$, then $S^\perp$ stands for the orthogonal complement of S in $L^2(Q)$. Note that in (H1) we can replace $u(x) \geq 1$ by $u(x) \geq a > 0$. In this case γ in (3.4) must be replaced by $a^{-1}\gamma$ and $\gamma < \alpha$ in Theorem 3.1 by $a^{-1}\gamma < \alpha$.

2. The linear operator associated with equation (1.1)

We shall use the complete orthonormal system of eigenfunctions $\{\psi_m \varphi_n; m \in \mathbb{Z}, n \in \mathbb{N}\}$ as a basis for $L^2(Q)$, where [10, p. 88]

$$\psi_m(t) = T^{-\frac{1}{2}} e^{i\mu_m t}, \quad \mu_m = 2m\pi T^{-1}, \quad m \in \mathbb{Z}, \tag{2.1}$$

and λ_n, φ_n are given by the Sturm-Liouville problem

$$-(u\varphi_n')_x = u\lambda_n^2 \varphi_n, \quad \varphi_n(0) = \varphi_n(\pi) = 0, \quad n \in \mathbb{N}, \tag{2.2}$$

with $\varphi' = \varphi_x = \frac{d\varphi}{dx}$.

It is known that λ_n is (increasingly) convergent to $+\infty$ as $n \to +\infty$. The inner product in $L^2(0, \pi)$ is defined by

$$\langle \eta, \theta \rangle = \int_0^\pi u(x) \eta(x) \bar{\theta}(x)\, dx, \quad \eta, \theta \in L^2(0, \pi). \tag{2.3}$$

Accordingly, $|\varphi_n|^2_{L^2(0,\pi)} = \int_0^\pi u(x) \varphi_n^2(x)\, dx = 1$.

Lemma 2.1. *Let u satisfy* (H1). *Then the eigenvalues λ_n^2 of problem* (2.2) *have the form*

$$\begin{aligned} &\lambda_n = n + \theta_n \quad \text{with } \theta_n \to 0 \text{ as } n \to +\infty, \\ &0 < \frac{b}{n} \le \sqrt{n^2+\rho} - n \le \theta_n \le \sqrt{n^2+\rho_1} - n \le \sqrt{1+\rho_1} - 1, \end{aligned} \tag{2.4}$$

where

$$b = \sqrt{1+\rho} - 1, \quad \rho_1 = \frac{2}{\pi}\int_0^\pi \eta_u(x)\,dx, \tag{2.5}$$

with η_u as defined in (1.3)

The proof of this lemma is based on the following fact.

Lemma 2.2. *Let η be a real function in $L^2(0,\pi)$ and let $\lambda_1^2 < \lambda_2^2 < \cdots$ and $z_1, z_2, \ldots$ denote the eigenvalues and real eigenfunctions of the Sturm-Liouville problem*

$$z_n''(x) + (\lambda_n^2 - \eta(x))z_n(x) = 0, \quad z_n(0) = z_n(\pi) = 0. \tag{2.6}$$

Then the following inequalities hold:

$$n^2 + \rho \le \lambda_n^2 \le \lambda + n^2 + \frac{2}{\pi}\int_0^\pi (\lambda - \eta(x))\,dx \tag{2.7}$$

for all $\lambda < \lambda_n^2, n = 1, 2, \ldots,$ where ρ is given in (1.2) *and*

$$(h(x))_+ = \max(h(x), 0); \quad (h(x))_- = (h(x))_+ - h(x).$$

Proof. By the substitution $z_n(x) = (u(x))^{1/2}\varphi_n(x)$, problem (2.2) is equivalent to (2.6) with $\eta = \eta_u(x)$ given by (1.3). In view of (1.2) one can prove that $\lambda_n^2 > \rho$ for all $n = 1, 2, \ldots$. (Indeed, assume for a contradiction that $\lambda_1^2 \le \rho$. Then $\lambda_1^2 \le \eta(x)$ for a.e. $x \in (0,\pi)$. Multiplying (2.6) by z_1 and integrating over $(0,\pi)$, one concludes that $z_1 \equiv 0$, which is absurd.) Therefore, we have $0 < \rho < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_n^2 < \cdots$. The lower bound for λ_n^2 in (2.8), i.e., $n^2 + \rho \le \lambda_n^2$ for all $n \in \mathbb{N}$, is easy to prove. Indeed, z_n has exactly $n-1$ zeros in $(0,\pi)$, denoted say by $0 < a_1 < a_2 < \cdots < a_{n-1} < \pi, 0 = a_0, \pi = a_n$. In view of (2.6) we have

$$\begin{aligned} &(\lambda_n^2 - \rho)\int_{a_{i-1}}^{a_i} z_n^2(x)\,dx = \int_{a_{i-1}}^{a_i} (z_n'(x))^2 + (\eta(x) - \rho)z_n^2(x)\,dx, \\ &\int_{a_{i-1}}^{a_i} z_n^2(x)\,dx \le \pi^{-2}(a_i - a_{i-1})^2 \int_{a_{i-1}}^{a_i} (z_n'(x))^2 dx, \quad i = 1, \ldots, n, \end{aligned} \tag{2.8}$$

which yields

$$\int_{a_{i-1}}^{a_i} (z_n'(x))^2 dx \le (\lambda_n^2 - \rho)\pi^{-2}(a_i - a_{i-1})^2 \int_{a_{i-1}}^{a_i} (z_n'(x))^2 dx;$$

so, dividing by $\int_{a_{i-1}}^{a_i} (z_n')^2 dx (a_i - a_{i-1})^2$ and summing over i, we get

$$\sum_{i=1}^n \frac{1}{(a_i - a_{i-1})^2} \le \pi^{-2}(\lambda_n^2 - \rho)n. \tag{2.9}$$

But $\min\{\sum_{i=1}^n \frac{1}{x_i^2}; x_i > 0, x_1 + \cdots + x_n = \pi\} = n(\frac{n}{\pi})^2$ and it is assumed for $x_1 = x_2 = \cdots = x_n = \frac{\pi}{n}$, so (2.9) implies $n^2 \le \lambda_n^2 - \rho$. The upper bound for λ_n^2 in (2.7) is taken from Theorem 3.2 in [6] (with $a = 0, b = \pi, q(x) = \eta(x)$, and λ_{n+1}^2 in place of λ_n). □

Proof of Lemma 2.1. For $\lambda = 0$, (2.7) becomes $n^2 + \rho \le \lambda_n^2 \le n^2 + \rho_1$, which leads to $\lambda_n = n + \theta_n$ with $\sqrt{n^2+\rho} - n \le \theta_n \le \sqrt{n^2+\rho_1} - n$. This proves that $\theta_n > 0$, $\theta_n \to 0$ as $n \to +\infty$, and (2.4) holds. The proof is complete. □

Definition 2.1. A function $y \in L^2(Q)$ is said to be a weak solution of the problem

$$\begin{aligned} &uy_{tt} - (uy_x)_x = f(x,t) \quad \text{in } Q, \quad f \in L^2(Q), \\ &y(0,t) = y(\pi,t) = 0, \quad t \in [0,T], \\ &y(x,0) = y(x,T), \quad y_t(x,0) = y_t(x,T), \quad x \in [0,\pi], \end{aligned} \tag{2.10}$$

if

$$\int_Q y(u\varphi_{tt} - (u\varphi_x)_x)\,dx\,dt = \int_Q f\varphi\,dx \tag{2.11}$$

for all $\varphi \in C_\pi^2(\bar{Q})$, where

$$C_\pi^2(\bar{Q}) = \{\varphi \in C^2(\bar{Q}), \varphi(0,t) = \varphi(\pi,t) = 0, \varphi(x,0) = \varphi(x,T), \varphi_t(x,0) = \varphi_t(x,T)\}. \tag{2.12}$$

Conversely, a weak solution of class $H^2(Q)$ satisfies (2.10) in classical sense.

Set

$$D(\tilde{A}) = \{y \in L^2(Q);\ \text{there is } f \in L^2(Q) \text{ such that (2.11) holds}\}. \tag{2.13}$$

Define $\tilde{A} : D(\tilde{A}) \to L^2(Q)$ and A by

$$\tilde{A}y = f, \quad y \in D\tilde{(A)}; \quad A = u^{-1}\tilde{A}. \tag{2.14}$$

Clearly, $D(A) = D(\tilde{A})$ contains the null function of $L^2(Q)$, and for each $y \in D(A)$ there is precisely one $f \in L^2(Q)$ satisfying (2.11) (due to the density of $C_\pi^2(\bar{Q})$ in $L^2(Q)$). Therefore the operator A defined by (2.13)–(2.14) is a linear operator from $L^2(Q) \to L^2(Q)$, and (2.11) can be rewritten as

$$\int_Q yA_0\varphi dxdt = \int_Q \tilde{A}y\varphi dxdt = \int_Q uAy\varphi dxdt, \quad \varphi \in C_\pi^2(\bar{Q}), y \in D(A), \tag{2.15}$$

where

$$A_0\varphi = u\varphi_{tt} - (u\varphi_x)_x, \quad \varphi \in C_\pi^2(\bar{Q}).$$

The operator $\tilde{A}$ defined by (2.13) and (2.14) is said to be *the linear operator associated with* (1.1). In what follows the main properties of A are given (with T as in (1.10)).

Lemma 2.3. *Let u satisfy* (H1). *Then the null space $N(A)$ of A is finite dimensional and it is given by*

$$N(A) = \operatorname{span}\{\psi_m\varphi_n; m \in \mathbb{Z}, n \in \mathbb{N} \text{ with } \lambda_n = |\mu_m|\} = N(\tilde{A}). \tag{2.16}$$

If $\rho_1 < 2p^{-1} + p^{-2}$, then $N(A)$ is zero (*where ρ_1 is given in* (2.5)).

Proof. Let $y \in N(A)$, i.e., $Ay = 0$, and let y_{mn} be the Fourier coefficients of y in $L^2(Q)$, i.e.,

$$y = \sum_{m,n} y_{mn}\psi_m(t)\varphi_n(x), \qquad y_{mn} = \int_Q u(x)y(x,t)\varphi_n(x)\bar{\psi}_m(t)dxdt, \tag{2.17}$$

$$\int_Q yA_0(\varphi)\,dx = 0, \quad \forall\varphi \in C_\pi^2(\bar{Q}). \tag{2.18}$$

Substituting $\varphi = \varphi_n \bar{\psi}_m$ into (2.18), we see that

$$y \in N(A) \text{ if and only if } (\lambda_n^2 - \mu_m^2) y_{mn} = 0, \tag{2.19}$$

which implies (2.16). Moreover it is easy to check that the equality

$$\lambda_n = |\mu_m|, \text{ i.e., } n + \theta_n = \frac{2|m|\pi}{T}, \quad n \in \mathbb{N}, m \in \mathbb{Z}, \tag{2.20}$$

with $T = 2\pi \frac{p}{q}$ can be valid for at most a finite numbers of pairs (m, n) Indeed, (2.20) means

$$pn + p\theta_n = |m|q \tag{2.21}$$

with $0 < p\theta_n < 1$ for n sufficiently large, so (2.21) has at most a finite number of solutions (m, n). If $\rho_1 < 2p^{-1} + p^{-2}$, then on the basis of (2.4) it follows that

$$0 < p\theta_n \leq p(\sqrt{1 + \rho_1} - 1) < 1 \text{ for all } n \in \mathbb{N}, \tag{2.21$'$}$$

so (2.21) has no solutions (m, n). This completes the proof. Note that

$$N(A)^\perp = \text{span}\{\psi_m \varphi_n; m \in \mathbb{Z}, n \in \mathbb{N} \text{ with } \lambda_n \neq |\mu_m|\}. \tag{2.22}$$

□

The main result of this section is

Proposition 2.1. *Let T be a rational multiple of π and let u satisfy* (H1). *Then $R(A)$ is closed in $L^2(Q)$, A is self-adjoint and $A^{-1} \in L(R(A), R(A))$. Moreover, we have*

$$|A^{-1} f| \leq d^{-1} |f|, \quad \forall f \in R(A), \tag{2.23}$$

where $d = \inf\{|\lambda_n^2 - \mu_m^2|, \lambda_n \neq |\mu_m|\}$,

$$\begin{aligned} &\langle A^{-1} f, f \rangle \geq -\alpha^{-1} |f|^2, \quad \forall f \in R(A), \\ &\langle Ay, y \rangle \geq -\alpha^{-1} |Ay|, \quad y \in D(A), \end{aligned} \tag{2.24}$$

where $\alpha = \inf\{\mu_m^2 - \lambda_n^2; |\mu_m| > \lambda_n\}$,

$$|A^{-1} f|_{L^\infty(Q)} \leq C|f|, \quad \forall f \in R(A), \tag{2.25}$$

$$|A^{-1} f|_{H^1(Q)} \leq C|f|_{H^1(Q)}, \quad \forall f \in H^1(Q) \cap R(A), \tag{2.26}$$

$$R(A) = N(A)^\perp; \quad L^2(Q) = N(A) \oplus R(A) = N(\tilde{A}) \oplus u^{-1} R(\tilde{A}). \tag{2.27}$$

Proof. Let y_{mn} and f_{mn} be the Fourier coefficients of y and f respectively, with respect to the orthonormal system $\{\psi_m \varphi_m\}$ defined by (2.1) and (2.2), i.e.,

$$y = \sum_{m,n} y_{mn} \psi_m \varphi_n; \quad f = \sum_{m,n} f_{mn} \psi_m \varphi_n; \quad \sum_{m,n} f_{mn}^2 = \int_Q u f^2 = |f|^2. \tag{2.28}$$

It follows that $Ay = f$ (i.e., (2.11) holds) if and only if

$$(\lambda_n^2 - |\mu_m|^2) y_{mn} = f_{mn}, \quad m \in \mathbb{Z}, n \in \mathbb{N}. \tag{2.29}$$

This shows that a necessary condition for the equation $Ay = f$ to have a solution y is $f \in N(A)^\perp$, i.e., $f_{mn} = 0$ for all (m, n) such that $\lambda_n = |\mu_m|$. We now prove that this condition is also sufficient. In other words we will prove that the series

$$\sum_{\lambda_n \neq |\mu_m|} |y_{mn}|^2 \text{ with } y_{mn} = \frac{f_{mn}}{\lambda_n^2 - \mu_m^2}, \quad \lambda_n \neq |\mu_m|, \tag{2.30}$$

is convergent (i.e., $R(A) = N(A)^{\perp}$ and (2.27) holds). The key fact is

$$\inf_{\lambda_n \neq |\mu_m|} \{|\lambda_n^2 - |\mu_m|^2|\} = d > 0. \tag{2.31}$$

Indeed, $\lambda_n = n + \theta_n$ with $\theta_n \to 0$ as $n \to \infty$, $\theta_n \geq \frac{b}{n} > 0$ (as in Lemma 2.1) and $\mu_m = |m|\frac{q}{p}$, so

$$|\lambda_n^2 - |\mu_m|^2| = p^{-2}|pn - |m|q + p\theta_n|(pn + |m|q + \rho\theta_n) \tag{2.32}$$

with

$$(pn + |m|q + p\theta_n) \geq \delta(n + |m|), \quad \delta = \min\{p, q\}. \tag{2.33}$$

If $pn = |m|q$, then

$$|\lambda_n^2 - |\mu_m|^2| \geq n\theta_n \geq b. \tag{2.34}$$

If $pn \neq |m|q$, we derive

$$\inf_{\substack{\lambda_n \neq |\mu_m| \\ pn \neq |m|q}} |pn - |m|q + p\theta_n| \geq c > 0 \tag{2.35}$$

(for some $c > 0$) due to $\lim_{n\to\infty} \theta_n = 0$.

Inequalities (2.34) and (2.35) imply (2.31). We now have on the basis of (2.30) and (2.31)

$$\sum_{\lambda_n \neq |\mu_m|} |y_{mn}|^2 \leq \frac{1}{d^2} \sum_{m,n} |f_{mn}|^2 = \frac{1}{d^2}|f|^2,$$

so (2.23) holds.

Inequality (2.24) is immediate. Indeed, by (2.29)

$$\langle A^{-1}f, f\rangle_{L^2(Q)} = \sum_{\lambda_n \neq |\mu_m|} \frac{f_{mn}^2}{\lambda_n^2 - |\mu_m|^2} \geq \sum_{\lambda_n < |\mu_m|} \frac{f_{mn}^2}{\lambda_n^2 - |\mu_m|^2},$$

which yields (2.24). In addition, (2.29) implies that A is symmetric.

If $f \in H^1(Q) \cap R(A)$, then the distributional derivative y_x of $y = A^{-1}f$ is given by

$$y_x = \sum_{\lambda_n \neq |\mu_m|} y_{mn}\psi_m\varphi_n', \tag{2.36}$$

where $\{\varphi_n'\}$ is orthogonal in $L^2(0, \pi)$ and

$$|\varphi_n'|^2_{L^2(0,\pi)} = \int_0^1 u(x)(\varphi_n')^2 dx = -\int_0^1 \varphi_n(u\varphi_n')_x dx = \lambda_n^2. \tag{2.37}$$

Therefore

$$\begin{aligned} |y_x|^2_{L^2(Q)} &= \sum_{\lambda_n \neq |\mu_m|} \lambda_n^2 |y_{mn}|^2 = \sum_{\lambda_n \neq |\mu_m|} \frac{\lambda_n^2 |f_{mn}|^2}{(\lambda_n^2 - |\mu_m|^2)^2} \\ &\leq \frac{1}{d^2} \sum_{\lambda_n \neq |\mu_m|} \lambda_n^2 |f_{mn}|^2 = \frac{1}{d^2}|f_x|^2_{L^2(Q)}. \end{aligned} \tag{2.38}$$

Similarly

$$|y_t|^2_{L^2(Q)} \leq \frac{1}{d^2} \sum_{\lambda_n \neq |\mu_m|} |\mu_m|^2 f_{mn}^2 = \frac{1}{d^2}|f_t|^2_{L^2(Q)}, \tag{2.39}$$

so (2.26) is proved, too.

In order to prove that $y = A^{-1}f \in L^\infty(Q)$ one first checks that

$$\sum_{\substack{n=1\\ \lambda_n \neq |\mu_m|}}^{\infty} \frac{1}{|\lambda_n^2 - |\mu_m|^2|} \leq C, \tag{2.40}$$

where C is a constant independent of m.

On the other hand, by (2.31)

$$|\lambda_n^2 - |\mu_m|^2|^2 \geq d|\lambda_n^2 - |\mu_m|^2|, \quad \lambda_n \neq |\mu_m|,$$

so

$$\sum_{\substack{n=1\\ \lambda_n \neq |\mu_m|}}^{\infty} \frac{1}{|\lambda_n^2 - |\mu_m|^2|^2} \leq C/d.$$

It follows that

$$\sum_{\substack{n=1\\ \lambda_n \neq |\mu_m|}}^{\infty} \frac{|f_{mn}|}{|\lambda_n^2 - |\mu_m|^2|} \leq Cd^{-1} \sum_n |f_{mn}|^2.$$

Therefore the series of y is bounded in $L^\infty(Q)$, namely:

$$|y(x,t)| \leq \sum_m \sum_{\substack{n=1\\ \lambda_n \neq |\mu_m|}} \frac{|f_{mn}|}{|\lambda_n^2 - |\mu_m|^2|} \leq Cd^{-1}(\sum_m \sum_n |f_{mn}|^2) \leq Cd^{-1}|f|^2_{L^2(Q)}.$$

This implies (2.25).

Finally, $D(A)$ contains $C^2_\pi(\bar{Q})$, which is dense in $L^2(Q)$, A is symmetric and (2.27) holds. Therefore A is selfadjoint, and the proof is complete. □

3. The nonlinear equation

We are now in a position to give the main results on (1.1).

Definition 3.1. The function $y \in L^2(Q)$ is said to be a weak solution to (1.1) in Q if

$$\int_Q y A_0 \varphi \, dx\, dt + \int_Q g(y)\varphi \, dx\, dt = \int_Q f\varphi \, dx\, dt \tag{3.1}$$

for all $\varphi \in C^2_\pi(\bar{Q})$ as indicated in (2.18), with A_0 defined in (2.15).

The last assumption on f and g is

(H3) $f \in L^\infty(Q)$, and

$$g(-\infty) + \delta \leq u(x)(P(u^{-1}f))(x,t) \leq g(+\infty) - \delta, \quad \text{a.e. } (x,t) \in Q,$$

for some $\delta > 0$. Here $P : L^2(Q) \to N(A)$ is the projection operator on $N(A)$.

We are now in a position to state the main result of this paper, namely

Theorem 3.1. *Assume that T is a rational multiple of π as in* (1.9) *and that Hypotheses* (H1), (H2) *and* (H3) *are fulfilled with*

$$0 < \gamma < \alpha, \quad \alpha = \inf\{|\mu_m|^2 - \lambda_n^2; \lambda_n < |\mu_m|\}.$$

Then equation (1.1) *has at least one weak solution $y \in L^\infty(Q)$. This weak solution is unique modulo $N(A)$, i.e., if y and z are weak solutions of* (1.1)*, then $y - z \in N(A)$. If g is strictly increasing, then the weak solution of* (1.1) *is unique.*

Proof. Denote by $\tilde{G}$ the realization of g in $L^2(Q)$, i.e.,

$$(\tilde{G}(y))(x,t) = g(y(x,t)) \quad \text{a.e. } (x,t) \in Q, y \in L^2(Q), \tag{3.2}$$

and set $G = u^{-1}\tilde{G}$. In view of (H2), $G : L^2(Q) \to L^2(Q)$ is a (continuous and) monotone operator, i.e.,

$$\langle G(y) - G(z), y - z\rangle \geq 0, \quad \forall y, z \in L^2(Q), \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(Q)$ as defined by (1.7). Moreover,

$$|G(y) - G(z)|^2 \leq \gamma\langle G(y) - G(z), y - z\rangle, \quad \forall y, z \in L^2(Q). \tag{3.4}$$

Replacing g in (1.4) by $g + \varepsilon$ (with $\varepsilon > 0$) one obtains (on the basis of monotonicity of $y \to g(y) + \varepsilon y$)

$$|g(y) + \varepsilon y - (g(z) + \varepsilon z)| \leq (\gamma + \varepsilon)|y - z|, \quad \forall y, z \in \mathbb{R}, \tag{3.5}$$

and accordingly

$$\begin{aligned} &|(G + \varepsilon I)(y) - (G + \varepsilon I)(z)|^2 \\ &\quad \leq (\gamma + \varepsilon)\langle (G + \varepsilon)y - (G + \varepsilon)z, y - z\rangle \end{aligned} \tag{3.6}$$

for all $y, z \in L^2(Q)$.

On the basis of (3.5) and (3.6), the following additional properties of G hold:

$$\begin{aligned} &R(G + \varepsilon I) = L^2(Q), \\ &\langle (G + \varepsilon I)^{-1}y - (G + \varepsilon I)^{-1}z, y - z\rangle \geq (\gamma + \varepsilon)^{-1}|y - z|^2, \quad \forall y, z \in L^2(Q). \end{aligned} \tag{3.6$'$}$$

In terms of A and G, y is a weak solution to (1.1) in Q (i.e., (3.1) holds) if and only if

$$Ay + G(y) = u^{-1}f. \tag{3.7}$$

In order to take advantage of the invertibility of $G + \varepsilon I$ we will consider the following approximation of (3.7), namely

$$Ay + (G + \varepsilon I)y = u^{-1}f, \quad y \in L^2(Q), \tag{3.8}$$

which can be equivalently written (using an idea of Brézis [4]) as

$$A^{-1}v + (G + \varepsilon I)^{-1}(u^{-1}f + v) \in N(A), \quad v \in R(A), \tag{3.9}$$

where $A^{-1} \in L(R(A))$ is the operator defined in Proposition 2.1.

Indeed, if y satisfies (3.8) (i.e., (3.1)) and $y_1 \in N(A)$, $y_2 \in R(A)$ are the components of y in $N(A) \oplus R(A) = L^2(Q)$ (i.e., $y = y_1 + y_2$), then (3.8) yields $(G + \varepsilon I)y = u^{-1}f - Ay_2$. Set $Ay_2 = v \in R(A)$, i.e., $y_2 = A^{-1}v$. Then $y = (G + \varepsilon I)^{-1}(u^{-1}f - v) = y_1 + A^{-1}v$, which shows that (3.8) and (3.9) are equivalent.

On the other hand, (3.9) is equivalent to

$$A^{-1}v + (G + \varepsilon I)^{-1}(u^{-1}f + v) + \partial J(v) \ni 0, \quad v \in R(A), \tag{3.10}$$

where J is the indicator function of $R(A)$ (i.e., $J(v) = 0$ if $v \in R(A)$ and $J(v) = +\infty$ if $v \notin R(A)$), and ∂J is the subdifferential of J. Taking into account that $\partial J(v)$ is the cone of the normals to $R(A)$ at v, it follows that $\partial J(v) = N(A)$, for all $v \in R(A)$.

Finally, (2.24) shows that $A^{-1} + \alpha^{-1}I$ is monotone on $R(A)$, so (3.10) can be rewritten in the equivalent form

$$(A^{-1} + \alpha^{-1}I)v + G_\alpha(v) + \partial J(v) \ni 0, \quad v \in R(A), \tag{3.11}$$

with $G_\alpha v = (G+\varepsilon I)^{-1}(u^{-1}f+v) - \alpha^{-1}v$. In view of $(3.6)'$, G_α satisfies

$$\langle G_\alpha v_1 - G_\alpha v_2, v_1 - v_2\rangle \geq ((\gamma+\varepsilon)^{-1} - \alpha^{-1})|v_1 - v_2|^2, \quad v_1, v_2 \in R(A). \tag{3.12}$$

We now prove that (3.11) has a solution v_ε for each $\varepsilon < \alpha - \gamma$.

On the basis of (3.12), for $\varepsilon < \alpha - \gamma$, G_α is coercive and maximal monotone in $L^2(Q)$ (being continuous and monotone).

A key step now is to prove that the monotone operator $v \to A_\alpha v + \partial J(v)$ with $A_\alpha = A^{-1} + \alpha^{-1}I, D(A_\alpha) = R(A)$ and $\partial J = N(A)$ is maximal monotone in $L^2(Q)$, i.e., for every $h \in L^2(Q)$, the equation (inclusion)

$$v + A_\alpha v + \partial J(v) \ni h \tag{3.13}$$

has a solution $v \in R(A)$. Indeed, this equation is equivalent to

$$v + A_\alpha v = (I-P)h, \qquad v \in R(A), \tag{3.14}$$

which has a unique solution $v \in R(A)$ (as A_α is continuous and monotone from $R(A)$ into itself). It follows that $(A_\alpha + \partial J) + G_\alpha$ is maximal monotone in $L^2(Q)$. Moreover, as G_α is coercive, $A_\alpha + \partial J + G_\alpha$ is onto. Therefore (3.11) has a solution $v_\varepsilon \in R(A)$ which is a solution of (3.9). This means that there is $y_\varepsilon^1 \in N(A)$ such that $A^{-1}v_\varepsilon + (G+\varepsilon I)^{-1}(u^{-1}f + v_\varepsilon) = y_\varepsilon^1$. Set $y_\varepsilon^2 = A^{-1}v_\varepsilon$. Then $y_\varepsilon = y_\varepsilon^1 - y_\varepsilon^2$ is a solution of (3.8).

We now prove that the solution y_ε of (3.8), i.e.,

$$\varepsilon y_\varepsilon + Ay_\varepsilon + Gy_\varepsilon = u^{-1}f, \tag{3.15}$$

is bounded in $L^\infty(Q)$. To this aim, we note that by the assumption (H3)

$$u(x)(P(u^{-1}f))(x,t) \subset K \subset \operatorname{int} R(g), \quad \text{a.e.}\,(x,t) \in Q, \tag{3.16}$$

where K is a compact interval. Hence, there is $\xi = \xi(x,t)$ with $|\xi| \leq C$, such that

$$u(x,t)(P(u^{-1}f))(x,t) + \delta w = g(\xi), \quad \text{a.e.}\,(x,t) \in Q, \tag{3.17}$$

for all $\delta > 0$ sufficiently small and $|w| = 1$. Then, the monotonicity of g yields

$$(g(y_\varepsilon) - uP(u^{-1}f) - \delta w)(y_\varepsilon - \xi) \geq 0, \quad \text{a.e.}\,(x,t) \in Q, \tag{3.18}$$

with $g(y_\varepsilon(x,t)) = u(x,t)(Gy_\varepsilon)(x,t)$ (by (3,2)), so

$$\delta w y_\varepsilon \leq (g(y_\varepsilon) - uP(u^{-1}f))y_\varepsilon - \xi(g(y_\varepsilon) - g(\xi)), \quad \text{a.e.}\,(x,t) \in Q, \tag{3.19}$$

which implies (for $w = y_\varepsilon(x,t)/|y_\varepsilon(x,t)|$) that

$$\delta|y_\varepsilon|_{L^1(Q)} \leq \langle Gy_\varepsilon - P(u^{-1}f), y_\varepsilon\rangle + C|Gy_\varepsilon| + C_1 \tag{3.20}$$

for some positive constants C and C_1.

On the other hand, in view of $L^2(Q) = N(A) \oplus R(A)$ (in (2.27)), $u^{-1}f = P(u^{-1}f) + Ay_1$ with $y_1 \in D(A)$ and $uP(u^{-1}f) = g(z) = uG(z)$ (according to (3.16) and (3.2)) for some $z = z(x,t)$ in $L^\infty(Q)$. Therefore, (3.15) can be written as

$$\varepsilon y_\varepsilon + A(y_\varepsilon - y_1) + G(y_\varepsilon) - G(z) = 0 \tag{3.21}$$

with $G(z) = P(u^{-1}f)$ and

$$\begin{aligned} &\langle A(y_\varepsilon - y_1), u_\varepsilon - y_1\rangle \geq -\alpha^{-1}|A(y_\varepsilon - y_1)|^2, \\ &\langle G(y_\varepsilon) - G(z), y_\varepsilon - z\rangle \geq \gamma^{-1}|Gy_\varepsilon - Gz|^2, \end{aligned} \tag{3.22}$$

with $\gamma^{-1} > \alpha^{-1}$. An elementary (but not immediate) combination of (3.20)–(3.22) jointly with an inequality of the form

$$ab \le \varepsilon a^2 + (4\varepsilon)^{-1} b^2, \quad \forall \varepsilon > 0, \quad a, b \in R, \tag{3.23}$$

leads to the boundedness of $|Ay_\varepsilon|$ and $|Gy_\varepsilon|$. On the other hand, by (3.20) $\delta |y_\varepsilon|_{L^1(Q)}$ can be estimated in terms of $|Ay_\varepsilon|$ and $|Gy_\varepsilon|$. Indeed,

$$\begin{aligned}\langle Gy_\varepsilon - Gz, y_\varepsilon \rangle &= \langle -\varepsilon y_\varepsilon - A(y_\varepsilon - y_1), y_\varepsilon \rangle \le -\langle A(y_\varepsilon - y_1), y_\varepsilon \rangle \\ &\le \alpha^{-1} |A(y_\varepsilon - y_1)|^2 - \langle A(y_\varepsilon - y_1), y_1 \rangle \le C.\end{aligned}$$

Going back to (3.20), we get

$$|y_\varepsilon|_{L^1(Q)} \le C, \qquad \forall \varepsilon > 0. \tag{3.24}$$

It is now easy to prove that $|y_\varepsilon|_{L^\infty(Q)}$ is bounded. To this goal, write $y_\varepsilon = y_\varepsilon^1 + y_\varepsilon^2$ with $y_\varepsilon^1 \in N(A)$ and $y_\varepsilon^2 \in R(A)$. Since $Ay_\varepsilon = Ay_\varepsilon^2$ is bounded in $L^2(Q)$, y_ε^2 is bounded in $L^\infty(Q)$ (by (2.25)). Consequently $y_\varepsilon^1 = y_\varepsilon - y_\varepsilon^2$ is bounded in $L^1(Q)$, so its Fourier coefficients $y^1_{\varepsilon mn} = \int_Q y_\varepsilon^1(x,t) u(x) \varphi_n(t) \psi_m(t)\, dx\, dt$ are bounded as $|\varphi_n(x)| \le C, |\psi_m(t)| \le C$ for some C independent of m, n, x and t. Therefore $|y^1_{\varepsilon mn}| \le C |y_\varepsilon^1|_{L^1(Q)} \le C_1$. Taking into account that $N(A)$ is finite dimensional, it follows that y_ε^1 is bounded in $L^\infty(Q)$, and hence so is y_ε. We now show that $\{Ay_\varepsilon\}$ and $\{Gy_\varepsilon\}$ are Cauchy sequences in $L^2(Q)$. To this goal, set $z_{\varepsilon\lambda} = \varepsilon y_\varepsilon - \lambda y_\lambda$. Clearly, $z_{\varepsilon\lambda} \to 0$ in $L^2(Q)$ as $\lambda, \varepsilon \to 0$. On the other hand, from (3.15) we have

$$\langle A(y_\varepsilon - y_\lambda), y_\varepsilon - y_\lambda \rangle + \langle G(y_\varepsilon) - G(y_\lambda), y_\varepsilon - y_\lambda \rangle \le C |z_{\varepsilon\lambda}|. \tag{3.25}$$

An obvious combination of (3.25), (3.4) and (2.24) leads to

$$\gamma^{-1} |G(y_\varepsilon) - G(y_\lambda)|^2 \le C |z_{\varepsilon\lambda}| + \alpha^{-1} |A(y_\varepsilon - y_\lambda)|^2. \tag{3.26}$$

Finally (3.26) in conjunction with $A(y_\varepsilon - y_\lambda) = G(y_\lambda) - G(y_\varepsilon) - z_{\varepsilon\lambda}$ and $\gamma\alpha^{-1} < 1$ implies that $|G(y_\lambda) - G(y_\varepsilon)| \to 0$ as $\lambda, \varepsilon \to 0$, and therefore $A(y_\varepsilon - y_\lambda)$ is also a Cauchy sequence in $L^2(Q)$. The sequence $\{y_\varepsilon\}$ is bounded in $L^2(Q)$, so it contains a weakly convergent subsequence (denoted again by $\{y_\varepsilon\}$ for simplicity). Say $y_\varepsilon \rightharpoonup y$ (weakly) in $L^2(Q)$. Taking into account that G is maximal monotone in $L^2(Q)$ (being continuous and monotone) and that Gy_ε is strongly convergent in $L^2(Q)$, it follows that $G(y_\varepsilon) \to G(y)$ (strongly) in $L^2(Q)$. Finally, it follows that $y \in D(A)$, $Ay_\varepsilon \to Ay$, and, letting $\varepsilon \downarrow 0$, (3.15) implies (3.7).

We now can prove that actually $y_\varepsilon \to y$ strongly in $L^2(Q)$. Indeed, $Ay_\varepsilon^2 = Ay_\varepsilon \to Ay$ strongly in $L^2(Q)$ so $y_\varepsilon^2 = A^{-1}(Ay_\varepsilon)$ is also strongly convergent in $L^2(Q)$ (say $y_\varepsilon^2 \to y^2$. Then $y^2 \in R(A)$). As $y_\varepsilon^1 = y_\varepsilon - y_\varepsilon^2 \rightharpoonup y - y^2$ and $N(A)$ is finite dimensional, it follows that $y_\varepsilon^1 \to y - y^2 = y^1$ and $y^1 \in N(A)$. The conclusion is that $y_\varepsilon \to y$ strongly in $L^2(Q)$. On the other hand y_ε is bounded in $L^\infty(Q)$, so $y \in L^\infty(Q)$. Finally, if y, z are two weak solutions of (3.7), then

$$G(y) - G(z) = -(Ay - Az). \tag{3.27}$$

In view of (3.4) and (2.24), this yields

$$\gamma^{-1} |G(y) - G(z)|^2 = \gamma^{-1} |Ay - Az|^2 \le \alpha^{-1} |A(y - z)|^2$$

with $\gamma^{-1} > \alpha^{-1}$. Therefore $A(y - z) = 0$, i.e., $y - z \in N(A)$.

If g is strictly increasing then G is one-to-one, so $G(y) - G(z) = 0$ implies $y = z$. The proof is complete. □

Corollary 3.1. *Let $T = 2\pi \frac{p}{q}$, where p and q are relatively prime positive integers, and let* (H1), (H2) *and* (H3) *be fulfilled. If*

$$\rho_1 = \frac{2}{\pi}\int_0^\pi \eta_u(x)\,dx < 2p^{-1} + p^{-2} \quad \textit{and} \quad 0 < \gamma < 2p^{-1} + p^{-2} - \rho_1, \tag{3.28}$$

then equation (1.1) *has a unique weak solution $y \in L^\infty(Q)$.*

Proof. On the basis of Lemma 2.3, in this case $N(A)$ is the trivial space so, according to Theorem 3.1 the weak solution is unique.

We now prove that

$$\rho_1 < 2p^{-1} + p^{-2} \quad \text{implies} \quad \alpha \geq 2p^{-1} + p^{-2} - \rho_1. \tag{3.29}$$

Set $\mu = p(\sqrt{1+\rho_1} - 1)$. We have $\mu < 1$ and (for $|\mu_m| > \lambda_n$)

$$\mu_m^2 - \lambda_n^2 = \frac{1}{p^2}(|m|q - np - p\theta_n)(|m|q + np + p\theta_n) \tag{3.30}$$

with $|m|q > np + p\theta_n$ and $p\theta_n \leq \mu < 1$ (see (2.21)). Therefore $|m|q \geq np + 1$, which yields

$$\begin{aligned} \mu_m^2 - \lambda_n^2 \geq& \frac{1}{p^2}(1 - p\theta_n)(2p + 1 + p\theta_n) \\ \geq& \frac{1}{p^2}(1 - \mu)(2p + 1 + \mu) = 2p^{-1} + p^{-2} - \rho_1, \end{aligned} \tag{3.31}$$

and, in turn,

$$\alpha = \inf\{\mu_m^2 - \lambda_n^2, |\mu_m| > \lambda_n\} \geq 2p^{-1} + p^{-2} - \rho_1. \tag{3.32}$$

Therefore, it suffices to choose

$$0 < \gamma < 2p^{-1} + p^{-2} - \rho_1, \tag{3.33}$$

which completes the proof. □

Note that $u = 1$ (or more generally $\eta_u = 0$, i.e., $u = (c_1x + c_2)^2$) implies $\rho_1 = 0$; so for $p = 1$, (3.33) contains the well known condition $\gamma < 3$ (see [4]).

Remark 3.1. 1) A careful examination of the proof of Theorem 3.1 shows that the results of this section remain valid under the more general assumption that $g = g(x, y)$ is continuous and nondecreasing in $y \in \mathbb{R}, x \to g(x, y) \in L^\infty$, and

$$|g(x, r) - g(x, \bar{r})| \leq \gamma |r - \bar{r}|, \quad \forall r, \bar{r} \in \mathbb{R}, \text{ a.e. } x \in \mathbb{R}.$$

Assumption (H3) should be modified as follows:

$$u(x)(Pu^{-1}f)(x, t) \subset K \quad \text{a.e. } (x, t) \in Q,$$

where K is a compact interval included in $(g(x, -\infty), g(x, +\infty))$ a.e. $x \in (0, \pi)$.

2) Theorem 3.1 with $y \in L^2(Q)$ instead of $L^\infty(Q)$ and Proposition 2.1 (except for (2.25), i.e., $A^{-1}f \in L^\infty(Q)$) remain valid if T is an irrational multiple of π of the form $T = 2\pi\alpha, \alpha = \sqrt{p_0/q_0}$, where p_0 and q_0 are relatively prime positive integers such that

$$\frac{p_0}{q_0} \neq \left(\frac{m}{n}\right)^2, \quad m, n \in \mathbb{N}, \quad \text{and} \quad p_0\rho_1 < 1.$$

Indeed, in this case $N(A)$ is zero, as

$$\lambda_n^2 - \mu_m^2 = (n + \theta_n)^2 - m^2\frac{q_0}{p_0} = p_0^{-1}(p_0n^2 - q_0m^2 + 2p_0n\theta_n + p_0\theta_n^2)$$

and, by Lemma 2.1

$$p_0(2n\theta_n + \theta_n^2) \le p_0\Big(\frac{2n\rho_1}{n+\sqrt{n^2+\rho_1}} + \frac{\rho_1^2}{(n+\sqrt{n^2+\rho_1})^2}\Big) \le p_0\rho_1 < 1.$$

This implies $\lambda_n \neq \mu_m$ and $\inf_{m,n} |\lambda_n^2 - \mu_m^2| = d \ge 1 - p_0\rho_1 > 0$.

In this case (2.40) may not remain valid, so $A^{-1}f$ may not be in $L^\infty(Q)$.

In connection with the regularity of the weak solutions of (1.1) we have

Corollary 3.2. *In addition to the hypotheses of Theorem* 3.1 *assume that* $f_t \in L^2(Q)$ *and either*

1) $\gamma < d, d = \inf\{|\lambda_n^2 - \mu_m^2|, \lambda_n \neq |\mu_m|\}$, *or*
2) *for each* $r > 0$ *there is* $\rho_r > 0$ *such that*

$$|g(s_1) - g(s_2)| \ge \rho_r |s_1 - s_2|, \quad \forall s_1, s_2 \in [-r, r]. \tag{3.34}$$

Then the weak solutions y *of* (1.1) *are in* $H^1(Q)$. *In the case of* (3.34) *the weak solution of* (1.1) *is unique.*

Proof. 1) Let y be a weak solution of (1.1), i.e., (3.7) holds: $Ay + G(y) = \tilde f$ with $\tilde f = u^{-1}f$. For a sufficiently small h set

$$hy^h(x,t) = y(x,t+h) - y(x,t), \quad (x,t) \in Q \text{ with } t \text{ fixed in } (0,T).$$

But $y = y_1 + y_2$ with $y_1 \in N(A)$ and $y_2 \in R(A)$, so $y^h = y_1^h + y_2^h$ and $y_1^h \in N(A)$ is bounded in $L^\infty(Q)$ with respect to h (as $N(A)$ is finite dimensional). We have

$$Ay^h + G^h y = \tilde f^h; \quad (G^h y)(x,t) = h^{-1}(g(y(x,t+h)) - g(y(x,t))), \tag{3.35}$$

and $|G^h y| \le \gamma |y^h|, \langle G^h y, y^h\rangle \ge \gamma^{-1}|G^h y|^2$, which yields

$$|y_2^h| \le |A^{-1}|(|\tilde f^h| + \gamma|y^h|) \le d^{-1}(|f_t + C + \gamma|y_2^h|), \tag{3.36}$$

where $|A^{-1}| \le d^{-1}$ (by (2.23)) and

$$|\tilde f^h| + \gamma|y_1^h| \le C + |\tilde f_t|.$$

By hypotheses, $d^{-1}\gamma < 1$, so (3.36) yields the boundedness of y_2^h in $L^2(Q)$, and therefore y^h is bounded in $L^2(Q)$, which implies that $y_t \in L^2(Q)$. This and the fact that y is an weak solution (i.e., (3.1) holds) lead to the conclusion that y_x exists too (in the distributional sense) and $y_x \in L^2(Q)$. Here is the proof.

We have

$$Ay = \tilde f - G(y) = F, \quad \text{with } \tilde f_t \in L^2(Q), y_t \in L^2(Q)$$

and $(G(y))_t \in L^2(Q)$. Denote by P the (linear bounded) projection operator on $R(A)$ and by F^ε the usual regularization (mollifier) of F. Let y^ε be the solution of $Ay^\varepsilon = PF^\varepsilon$. Therefore $A(Py^\varepsilon) = PF^\varepsilon$, $PF^\varepsilon \to PF = F$, $A(Py^\varepsilon) \to F = Ay = APy$; so $Py^\varepsilon \to Py$ as $\varepsilon \downarrow 0$.

By the definition of the weak solution Py^ε

$$\int_Q u\varphi_x(Py_x^\varepsilon)\,dx\,dt = \int_Q u\varphi_t(Py_t^\varepsilon)\,dx\,dt + \int_Q \varphi(PF^\varepsilon)\,dx\,dt.$$

Replacing here $\varphi = Py^\varepsilon$ and taking into account (2.39), i.e., $Py_t^\varepsilon| \le \frac{1}{d}|PF_t^\varepsilon|$ (which is bounded in $L^2(Q)$ by the norm of $f_t - (G(y))_t$ plus a constant), it follows that

Py^ε_x is bounded in $L^2(Q)$. This and $Py^\varepsilon \to Py$ imply $Py_x \in L^2(q)$ in the sense of distributions. As $N(A)$ is finite dimensional and $(I-P)y \in N(A)$, $(I-P)y$ is in $H^1(Q)$. Thus $y_x \in L^2(q)$, so $y \in H^1(Q)$.

2) Condition (3.34) and $y \in L^\infty(Q)$ (i.e., $|y(x,t)| \le r$ a.e. $(x,t) \in Q$, for some $r > 0$) give

$$|h^{-1}||g(y(x,t+h)) - g(y(x,t))| \ge \rho_r |y^h(x,t)|, \quad \text{a.e. } (x,t) \in Q, \tag{3.37}$$

for all sufficiently small h (here t is arbitrary in $(0,T)$, but fixed). By the definition of G (e.g. (3.2)) and G^h (see (3.35)), it follows that

$$|G^h y| \ge \rho_r |y^h|, \quad \langle G^h y, y^h\rangle \ge \gamma^{-1}\rho_r^2 |y^h|^2. \tag{3.38}$$

Multiplying $Ay^h + G^h y = f^h$ by y^h, using $\langle Ay^h, y^h\rangle \ge -\alpha^{-1}|Ay^h|^2$, (3.38) and

$$\langle f^h, y^h\rangle \le \beta |y^h|^2 + \frac{1}{4\beta}|f^h|^2$$

with $0 < \beta < \gamma^{-1}\rho_r^2 - \alpha^{-1}\rho_r^2$, one obtains

$$(\gamma^{-1}\rho_r^2 - \beta)|y^h|^2 \le \alpha^{-1}|Ay^h|^2 + C \tag{3.39}$$

and

$$\gamma^{-1}|G^h y|^2 \le \alpha^{-1}|Ay^h|^2 + \beta|y^h|^2 + C,$$

which gives (via (3.39))

$$|G^h y|^2 \le d|Ay^h|^2 + C_1 \tag{3.39}$$

with $d = \alpha^{-1}\rho_r^2/(\gamma^{-1}\rho_r^2 - \beta)$ (so $d < 1$) and C, C_1 independent of h. Finally, substituting $G^h y = \tilde{f}^h - Ay^h$ into (3.39), we see obviously that $|Ay^h|$ is bounded in $L^2(Q)$ (with respect to all sufficiently small h). Going back to (3.39), one gets the boundedness of y^h in $L^2(Q)$. As seen in 1), this implies $y \in H^1(Q)$. The uniqueness of the weak solution follows from Theorem 3.1 and (3.34) (which implies that g is strictly increasing). The proof is complete. □

The case $\eta_u = 0$ (see (1.3)) remains an open problem.

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